Lösungsvorschlag 4

1) Unbestimmte Integrale

a) $\int x e^{-x^2} dx$

Substituiere $u(x) = x^2 \quad u'(x) = 2x \Rightarrow x = \frac{1}{2}u'(x)$

$$= \int \frac{1}{2} u'(x) e^{-u(x)} dx = -\frac{1}{2} e^{-u(x)} + C = -\frac{1}{2} e^{-x^2} + C$$

alternativ

$$= \int \frac{1}{2} e^{-u} du = -\frac{1}{2} e^{-u} + C = -\frac{1}{2} e^{-x^2}$$

alternativ mit Differentialen (formale Manipulation)

$$u(x) = x^2 \quad \frac{du}{dx} = 2x \Rightarrow du = 2x\,dx$$

$$= \int \frac{1}{2} e^{-x^2} (2x\,dx) = \int \frac{1}{2} e^{-u} du = -\frac{1}{2} e^{-u} + C = -\frac{1}{2} e^{-x^2} + C$$

b) $\int \sqrt{1-x^2}\, dx$ lässt sich mit einer umgekehrten Substitution lösen, raten anhand der Struktur oder dem angegebenen Endergebnis. Versuche $x(t) = \sin t$

$x'(t) = \cos t$ $\qquad$ (oder $\frac{dx}{dt} = x'(t) \Rightarrow dx = x'(t)dt$)

$$\int \sqrt{1-x^2}\, dx = \int \sqrt{1-x(t)^2}\, x'(t)\, dt$$

$$= \int \sqrt{1-\sin^2(t)} \cos t\, dt = \int \cos^2 t\, dt$$

Das lässt sich mittels partieller Integration berechnen:

$$\int \cos^2 t\, dt = \sin(t)\cos(t) + \int \sin^2 t\, dt = \sin t \cos t + \int (1-\cos^2 t) dt$$

$$= \sin t \cos t + t - \int \cos^2(t)\, dt + C$$

Das kann man nach dem gesuchten Integral auflösen:

$$2\int \cos^2 t\, dt = \sin t \cos t + t + C$$

$$\int \cos^2 t\, dt = \frac{1}{2}\sin t \cos t + \frac{t}{2} + C'$$

Rücksubstitution: $\cos t = \sqrt{1-\sin^2 t} = \sqrt{1-x^2}$

$\sin t = x \Rightarrow t = \arcsin x$

$$\Rightarrow \int \sqrt{1-x^2}\, dx = \frac{1}{2} x \sqrt{1-x^2} + \frac{1}{2}\arcsin x + C' \quad \text{q.e.d.}$$

c) $\int \frac{x}{\sqrt{1-x^4}}\, dx$ Substitution $u(x) = x^2$ $u'(x) = 2x = \frac{du}{dx}$

$$= \frac{1}{2}\int \frac{u'(x)}{\sqrt{1-u(x)^2}}\, dx = \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\, du = \frac{1}{2}\arcsin u + C$$

$$= \frac{1}{2}\arcsin(x^2) + C$$

d) $\int \frac{\sin(2x)}{3+\sin^2(x)}\, dx$ Verwende $\sin(2x) = 2\sin x \cos x$

$$= 2\int \frac{\sin x \cos x}{3+\sin^2 x}\, dx$$ Substitution $u(x) = \sin(x)$

$u'(x) = \cos x = \frac{du}{dx}$

$$= 2\int \frac{u(x)\, u'(x)}{3+u^2(x)}\, dx = \int \frac{2u}{3+u^2}\, du$$

weitere Substitution $v = u^2$ $v'(u) = 2u = \frac{dv}{du}$

$$= \int \frac{v'(u)}{3+v}\, du = \int \frac{1}{3+v}\, dv = \ln|3+v| + C$$

$$= \ln|3+u^2| + C = \ln|3+\sin^2(x)| + C$$

2) a) $\int x^2 e^{2x} dx$

Durch partielle Integration

$$\int \underset{u}{x^2} \underset{v'}{e^{2x}} dx = \underset{u}{x^2} \underset{v}{\frac{1}{2}e^{2x}} - \int \underset{u'}{2x} \underset{v}{\frac{1}{2}e^{2x}} dx$$

$$= \frac{x^2}{2}e^{2x} - \int \underset{u}{x} \underset{v'}{e^{2x}} dx = \frac{x^2}{2}e^{2x} - \left[\underset{u}{x} \underset{v}{\frac{1}{2}e^{2x}} - \int \underset{u'}{1} \cdot \underset{v}{\frac{1}{2}e^{2x}} dx \right]$$

$$= \frac{x^2}{2}e^{2x} - \frac{x}{2}e^{2x} + \int \frac{1}{2}e^{2x} dx$$

$$= \frac{x^2}{2}e^{2x} - \frac{x}{2}e^{2x} + \frac{1}{4}e^{2x} + C$$

$$= e^{2x}\left[\frac{x^2}{2} - \frac{x}{2} + \frac{1}{4}\right] + C.$$

b) $\int \underset{u'}{x^3} \underset{v}{\ln(x)} dx$ durch partielle Integration

$$= \underset{u}{\frac{1}{4}x^4} \underset{v}{\ln(x)} - \int \underset{u}{\frac{1}{4}x^4} \underset{v'}{\frac{1}{x}} dx = \frac{1}{4}x^4 \ln(x) - \frac{1}{4}\int x^3 dx$$

$$= \frac{1}{4}x^4 \ln(x) - \frac{1}{4}\cdot\frac{1}{4}x^4 + C = \frac{1}{4}x^4\left(\ln(x) - \frac{1}{4}\right). + C$$

3) Partialbruchzerlegung

a) $\int \frac{3x}{x^3+3x^2-4} dx$

Der Grad des Nenners ist größer als der Grad des Zählers, daher brauche ich keine Polynomdivision durchzuführen.

Faktorisieren des Nenners? Nullstellen raten! $x=1$ funktioniert:

$1+3-4=0 \Rightarrow x=1$ ist Nullstelle, daher lässt sich $(x-1)$ ausklammern:

Polynomdivision:

$$(x^3+3x^2-4):(x-1) = x^2+4x+4 = (x+2)^2$$
$$\begin{array}{l} x^3 - x^2 \\ \quad 4x^2 - 4 \\ \quad \underline{4x^2 - 4x} \\ \qquad 4x-4 \\ \qquad \underline{4x-4} \\ \qquad\ 0 \end{array}$$

$$(x^3+3x^2-4) = (x-1)(x+2)^2$$

Ansatz für die Partialbruchzerlegung

$$\frac{3x}{x^3+3x^2-4} = \frac{3x}{(x+2)^2(x-1)} = \frac{a}{x-1} + \frac{b}{(x+2)} + \frac{c}{(x+2)^2}$$

$$= \frac{a(x+2)^2 + b(x-1)(x+2) + c(x-1)}{(x+2)^2(x-1)}$$

$$a(x^2+4x+4) + b(x^2+x-2) + c(x-1)$$
$$= (a+b)x^2 + (4a+b+c)x + 4a-2b-c \overset{!}{=} 3x$$

Gleichungssystem aus dem Koeffizientenvergleich

$$\begin{array}{llllll} (1) & a & +b & & = 0 \\ (2) & 4a & +b & +c & = 3 \\ (3) & 4a & -2b & -c & = 0 \end{array}$$

$$\begin{array}{llllll} (1) & a & +b & & = 0 \\ (2)-4\cdot(1) & 0 & -3b & +c & = 3 \\ (3)-4\cdot(1) & 0 & -6b & -c & = 0 \end{array}$$

$$\begin{array}{llllll} (1) & a & +b & & = 0 \\ (2) & & -3b & +c & = 3 \\ (3)-2\cdot(2) & & 0 & -3c & = -6 \end{array}$$

(3) $c = \frac{6}{3} = 2$

(2) $-3b = 3 - c \qquad b = \frac{1}{3}(c-3) = \frac{-1}{3}$

(1) $a = -b = \frac{1}{3}$

$$f(x) = \frac{1}{3}\frac{1}{x-1} - \frac{1}{3}\frac{1}{(x+2)} + \frac{2}{(x+2)^2}$$

$$F(x) = \int dx\, f(x) = \frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|(x+2)| - \frac{2}{(x+2)} + C$$

b) $\int \frac{x-4}{x^3+x}\,dx$ Grad des Nenners ist größer als Grad des Zählers, Polynomdivision nicht notwendig.

Nenner $(x^2+1)x = x^3+x$

Ansatz für die Partialbruchzerlegung:

$$\frac{a}{x} + \frac{bx+c}{x^2+1} \overset{!}{=} \frac{x-4}{x(x^2+1)}$$

x^2+1 ist ein Polynom mit nicht-reellen Nullstellen $z_0 = \pm i$ $\rightarrow$ Ansatz $bx+c$ im Zähler

$$a(x^2+1) + x(bx+c) \overset{!}{=} x-4$$

$$x^2(a+b) + cx + a = x-4$$

$$\begin{aligned} a + b &= 0 \\ c &= 1 \\ a &= -4 \end{aligned} \qquad \begin{aligned} a &= -4 \\ a + b &= 0 \\ c &= 1 \end{aligned}$$

$\Rightarrow$ $a = -4$

$b = -a = 4$

$c = 1$

$$f(x) = -\frac{4}{x} + \frac{4x+1}{x^2+1}$$

$$\int f(x)\,dx = \int\left(-\frac{4}{x}\right)dx + 2\int \frac{2x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx$$

$$= -4\ln|x| + 2\ln|x^2+1| + \arctan x + C$$

$$= -2\ln|x|^2 + 2\ln|x^2+1| + \arctan x + C$$

$$= 2\ln\left|\frac{x^2+1}{x^2}\right| + \arctan x + C$$

4) Konvergenz uneigentlicher Integrale

a) $\displaystyle\int_0^\infty \frac{1}{\sqrt{x}\,(1+x)}\,dx$ Substitution $u(x) = \sqrt{x}$, $u'(x) = \dfrac{1}{2\sqrt{x}}$

$$\int_0^\infty 2u'(x)\frac{1}{(1+u^2(x))}\,dx = 2\int_0^\infty \frac{du}{(1+u^2)} = 2\arctan(u)\Big|_0^\infty$$

$$= 2\frac{\pi}{2} - 0 = \pi$$

b) $$\int_0^1 \ln(x)\,dx = x\ln(x)\Big|_0^1 - \int_0^1 x\frac{1}{x}\,dx$$

$$= x\ln(x)\Big|_0^1 - [x]_0^1$$

$$= 0 - \lim_{x\to 0^+}(x\ln(x)) - 1$$

$$x\ln(x) = \frac{\ln x}{1/x} \xrightarrow[x\to 0]{} \text{"}\frac{\infty}{\infty}\text{"}$$ $f(x) = \ln(x)$, $f'(x) = \frac{1}{x}$, $g(x) = \frac{1}{x}$, $g'(x) = -\frac{1}{x^2}$

$$\lim_{x\to 0}\frac{f'(x)}{g'(x)} = \lim_{x\to 0}\frac{-x^2}{x} = \lim_{x\to 0}(-x) = 0 = \lim_{x\to 0} x\ln(x)$$

5) Differenzieren von Integralen.

a) $$\int_2^{x^2} \frac{\cos^2 t}{1+\cos t}\,dt = F(x^2) - F(2)$$

$$\frac{d}{dx} F(x^2) = F'(x^2)\cdot 2x = f(x^2)\cdot 2x$$

$$= \frac{\cos^2(x^2)}{1+\cos(x^2)}\cdot 2x$$

b) $$\int_{-x}^{x} e^{-t^2}\,dt = F(x) - F(-x)$$

$$\frac{d}{dx}\left(F(x) - F(-x)\right) = F'(x) - F'(-x)(-1)$$

$$= f(x) + f(-x) = e^{-x^2} + e^{-x^2} = 2e^{-x^2}.$$

6) Komplexe Zahlen

a) $$z = \frac{5+3i}{5+i}$$

Real + Imaginärteil?
Mit dem komplex Konjugierten des Nenners erweitern!

$$z = \frac{(5+3i)(5-i)}{(5+i)(5-i)}$$

$$= \frac{25+15i-5i+3}{5^2-(-1)} = \frac{(25+3)+i10}{26} = \frac{14}{13} + i\frac{5}{13}$$

$$\mathrm{Re}(z) = \frac{14}{13} \qquad \mathrm{Im}(z) = \frac{5}{3}$$

$$= \frac{z+z^*}{2} \qquad = \frac{1}{2i}(z-z^*)$$

b) $z = 2 + 2i$ in Polardarstellung.

Erinnerung: $z = r e^{i\varphi} = r(\cos\varphi + i\sin\varphi)$

$$\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)} = \frac{\sin\varphi}{\cos\varphi} = \tan\varphi \Rightarrow \varphi = \arctan\left(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right)$$

$$z z^* = r^2 = (\mathrm{Re}(z))^2 + (\mathrm{Im}(z))^2$$

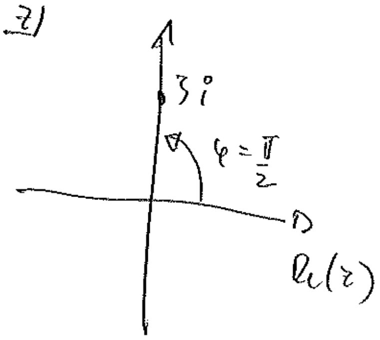
hier: $r^2 = 2^2 + 2^2 = 4 + 4 = 8 \quad r = \sqrt{8} = 2\sqrt{2}$

$\varphi = \arctan\left(\frac{2}{2}\right) = \arctan(1) = \frac{\pi}{4}$

$\Rightarrow z = 2\sqrt{2}\, e^{i\frac{\pi}{4}}$

c) $z = 3e^{i\frac{\pi}{2}} = 3(\underbrace{\cos\tfrac{\pi}{2}}_{0} + i\underbrace{\sin\tfrac{\pi}{2}}_{1}) = i3$

(Für $\varphi = \frac{\pi}{2}$ liegt die komplexe Zahl auf der imaginären Achse)



d) Lösung der Gleichung $z^2 + 2z - i = 0$

Direkte Lösung: $z_{1,2} = -1 \pm \sqrt{1+i}$

Wurzelziehen: $1 + i = \sqrt{2}\, e^{i\frac{\pi}{4}}$

$r^2 = z z^* = 1^2 + i(-i) = 2 \qquad \frac{\mathrm{Im}(z)}{\mathrm{Re}(z)} = \frac{1}{1} \qquad \arctan(1) = \frac{\pi}{4}$

Generell: Lösung einer Gleichung vom Typ $z^n = z_0$

$$(r e^{i\varphi})^n = r_0 e^{i\varphi_0} = r^n e^{i\varphi n} \underbrace{e^{i2\pi k}}_{=1} \quad k \in \mathbb{Z}$$

Dann $r_0 = r^n \rightarrow r = \sqrt[n]{r_0}$

$\varphi_0 = \varphi_n + 2\pi k \Rightarrow \varphi = \frac{1}{n}(\varphi_0 - 2\pi k) \quad k = 0, 1, 2, \ldots, n-1$

liefert n Wurzeln der Gl.

hier: $\left(\sqrt{2}\, e^{i\frac{\pi}{4}+i2\pi k}\right)^{1/2} = 2^{1/4}\, e^{i\frac{\pi}{8}+i2\pi\frac{k}{2}}$ (*)

$$= 2^{1/4}\, e^{i\pi\left(\frac{1}{8}+k\right)} \qquad k=0,1$$

$$= \begin{cases} 2^{1/4}\, e^{i\frac{\pi}{8}} & k=0 \\ 2^{1/4}\, e^{i\pi\frac{9}{8}} & k=1 \end{cases}$$

$$z_{1,2} = -1 \pm 2^{1/4} e^{i\frac{\pi}{8}}$$

benutze

$(-1) = e^{+i\pi}$

$-e^{i\frac{\pi}{8}} = e^{+i\pi}\, e^{i\frac{\pi}{8}} = e^{i\pi\frac{9}{8}}$

$\Rightarrow$ $\pm e^{i\frac{\pi}{8}}$ entspricht $\begin{cases} e^{i\pi/8} \\ e^{i\pi 9/8} \end{cases}$

Elegantere Lösung:
Quadratisch ergänzen

$$z^2 + 2z + 1 - 1 - i = 0$$
$$(z+1)^2 = 1+i = \sqrt{2}\, e^{i\frac{\pi}{4}}$$

Wurzel ziehen:

$$z+1 \overset{(*)}{=} 2^{1/4}\, e^{i\frac{\pi}{8}(1+8k)} \qquad k=0,1$$

$$= 2^{1/4} \begin{cases} e^{i\frac{\pi}{8}} \\ e^{i\pi\frac{9}{8}} \end{cases}$$

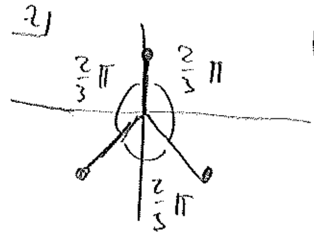
$$z_1 = 2^{1/4} e^{i\pi/8} - 1 \quad , \quad z_2 = 2^{1/4}\, e^{i\pi 9/8} \qquad \text{wie oben.}$$

e) Lösungen der Gleichung $z^3 + i = 0$ $\qquad z^3 = -i = e^{i\frac{3}{2}\pi}\, e^{i2\pi k}$

$$z = \left(e^{i\frac{3}{2}\pi + i2\pi k}\right)^{1/3} = e^{\underbrace{i\frac{\pi}{2} + i\frac{2}{3}\pi k}} = e^{i\frac{\pi}{6}(3+4k)}$$

$k = 0,1,2$

$$z_1 = e^{i\frac{\pi}{2}} = i$$
$$z_2 = e^{i\frac{\pi}{6}(7)} = e^{i\pi\frac{7}{6}}$$
$$z_3 = e^{i\frac{\pi}{6}11}$$



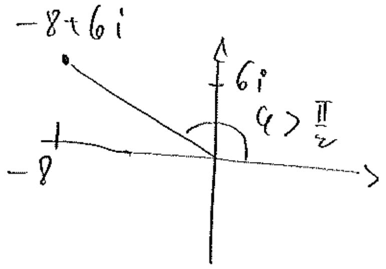
Winkeldifferenz bei der dritten Wurzel $\frac{2\pi}{3}$

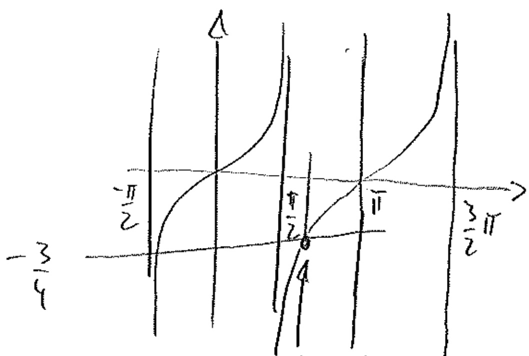
f) $\ln(-8+6i)$

Polardarstellung $r^2 = zz^* = 8^2 + 6^2 = 64 + 36 = 100$

$r = 10$

Polarwinkel: $\tan\varphi = \frac{\mathrm{Im}(z)}{\mathrm{Re}(z)} = -\frac{6}{8} = -\frac{3}{4} \approx -0.75$





gesuchtes φ ist hier $\frac{\pi}{2} < \varphi < \pi$

$\arctan\left(-\frac{3}{4}\right) \approx -0.6435$

$\varphi = \pi - \arctan\left(-\frac{3}{4}\right) \approx +2.5$

$-8+6i = 10\, e^{i\varphi}$ mit $\varphi \approx -2.49809 \approx 2.5$

$\ln\left(10\, e^{i\varphi + i2\pi k}\right) = \ln(10) + \ln\left(e^{i\varphi + i2\pi k}\right)$

$= \ln(10) + i\varphi + i2\pi k \qquad k \in \mathbb{Z}$

$\approx 2.3 + i2.5 + i2\pi k$ („Zweige“ des komplexen Logarithmus.

Hauptwert von $\ln(z) = \ln(re^{i\varphi}) = \ln(r) + i\varphi$ mit $0 \le \varphi < 2\pi$

heißt $\mathrm{Ln}(z)$

$\mathrm{Ln}(-8+6i) = \ln(10) + i\varphi \approx 2.3 + i2.5.$

7* a) $$\int \frac{e^{3x}}{e^{2x}-1}\,dx \qquad u(x) = e^x \quad u'(x) = e^x$$

$$= \int \frac{u(x)^2\, u'(x)}{u^2(x)-1}\,dx = \int \frac{u^2}{u^2-1}\,du$$

Polynom gleichen Grades in Zähler + Nenner: Polynom dividieren.

$$\frac{u^2-1+1}{u^2-1} = 1 + \frac{1}{u^2-1} \qquad u^2-1 = (u-1)(u+1)$$

daher Ansatz für Partialbruchzerlegung

$$= 1 + \frac{a}{u-1} + \frac{b}{u+1} = 1 + \frac{a(u+1)+b(u-1)}{(u-1)(u+1)}$$

Koeffizientenvergleich: $a+b=0 \qquad a-b=1$

$$\begin{array}{r} a+b=0 \\ a-b=1 \\ \hline 2a \quad =1 \end{array} \qquad a=\frac{1}{2},\; b=-\frac{1}{2}$$

$$\frac{u^2}{u^2-1} = 1 + \frac{1}{2}\left(\frac{1}{u-1} - \frac{1}{u+1}\right) \quad ✓$$

$$\int \frac{u^2}{u^2-1}\,du = u + \frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1| + C$$

$$= u + \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C$$

$$\int \frac{e^{3x}}{e^{2x}-1}\,dx = e^x + \frac{1}{2}\ln\left|\frac{e^x-1}{e^x+1}\right| + C.$$

b) $\int \frac{x^7+1}{x^5+x^3}\,dx$ Zählergrad > Nennergrad, Polynomdivison.

$$(x^7+1):(x^5+x^3) = x^2-1+\frac{x^3+1}{x^3(x^2+1)}$$

$$\begin{array}{l} x^7+x^5 \\ \hline -x^5+1 \\ -x^5-x^3 \\ \hline x^3+1 \end{array}$$

Ansatz für Partialbruchzerlegung: $\frac{x^3+1}{x^3(x^2+1)} = \frac{a}{x}+\frac{b}{x^2}+\frac{c}{x^3}+\frac{d_1x+d_2}{x^2+1}$

$$= \frac{(ax^2+bx+c)(x^2+1)+x^3(d_1x+d_2)}{x^3(x^2+1)} = \frac{x^3+1}{x^3(x^2+1)}$$

Koeffizientenvergleich

$$(a+d_1)x^4+(b+d_2)x^3+(a+c)x^2+bx+c = x^3+1$$

$a+d_1 = 0$

$b+d_2 = 1$

$a+c = 0$

$b = 0 \Rightarrow d_2 = 1$

$c = 1 \Rightarrow a = -1 \Rightarrow d_1 = 1$

$$\frac{x^7+1}{x^5+x^3} = x^2-1-\frac{1}{x}+\frac{1}{x^3}+\frac{x+1}{x^2+1}$$

$\int \frac{1}{x}dx = \ln|x|$ $\quad \int \frac{1}{x^3}dx = -\frac{1}{2x^2}$ $\quad \int dx\,\frac{x}{x^2+1} = \frac{1}{2}\ln|x^2+1|$

$\int \frac{1}{x^2+1} = \arctan(x)$

$$F(x) = \frac{1}{3}x^3 - x - \ln|x| - \frac{1}{2x^3} + \frac{1}{2}\ln|x^2+1| + \arctan(x) + C$$

c) $\int \sqrt{1+x^2}\, dx$ partielle Integration:

$$\int \underset{v'}{1} \underset{u}{\sqrt{1+x^2}}\, dx = x\sqrt{1+x^2} - \int x \frac{1}{2} \frac{\cancel{2}x}{\sqrt{1+x^2}}\, dx$$

$$= x\sqrt{1+x^2} - \int \frac{x^2}{\sqrt{1+x^2}}\, dx$$

$x = \sinh(u)$ $\quad x'(u) = \cosh(u)$

$\cosh^2 u - \sinh^2 u = 1 \quad \Rightarrow \quad 1 + \sinh^2(u) = \cosh^2(u)$

$$\int \frac{x^2}{\sqrt{1+x^2}}\, dx = \int \frac{\sinh^2(u)}{\cosh(u)} \cosh(u)\, du =$$

$$= \int \sinh^2(u)\, du \overset{\text{Part. Int.}}{=} \cosh(u)\sinh(u) - \int \cosh^2(u)\, du$$

$$= \cosh(u)\sinh(u) - \int du\, (1 + \sinh^2(u))$$

$$= \cosh(u)\sinh(u) - u - \int du\, \sinh^2(u)$$

Auflösen nach gesuchtem Integral:

$$\int \sinh^2(u)\, du = -\frac{u}{2} + \frac{1}{2}\cosh(u)\sinh(u)$$

Rücksubstitution: $\quad u = \text{Arsinh}(x) \qquad \sinh(u) = x$

$$\cosh(u) = \sqrt{1+\sinh^2(u)} = \sqrt{1+x^2}$$

Damit insgesamt

$$F(x) = x\sqrt{1+x^2} - \left[-\frac{1}{2}\text{Arsinh}(x) + \frac{1}{2}\sqrt{1+x^2}\, x \right] + C$$

$$= \frac{1}{2} x\sqrt{1+x^2} + \frac{1}{2}\text{Arsinh}(x) + C \,.$$

a) $\int \frac{1}{1+\cosh(x)}$ $\qquad \cosh(x) = \frac{1}{2}(e^x + e^{-x})$

Substitution $u = e^x \quad u'(x) = e^x = u(x) \quad \frac{du}{dx} = u$

$$= \int \frac{1}{1+\frac{1}{2}(u+\frac{1}{u})} \frac{1}{u} du = \int \frac{1}{u+\frac{1}{2}u^2+\frac{1}{2}} du = \int \frac{2}{u^2+2u+1} du$$

$$= \int \frac{2}{(u+1)^2} du = -2 \frac{1}{1+u} + C = -\frac{2}{1+e^x} + C.$$

8) $\int_0^{\pi/2} \frac{dx}{\sin^2(x)}$ $\qquad$ benutze $\sin(x) \le x$ für $x \in [0, \pi/2]$

Dann $\int_0^{\pi/2} \frac{dx}{\sin^2(x)} \ge \int_0^{\pi/2} \frac{dx}{x^2} = \left[\left(-\frac{1}{x}\right)\right]_0^{\pi/2} \to \infty$ divergiert.

Daher divergiert auch $\int_0^{\pi/2} \frac{dx}{\sin^2(x)} \to \infty$.

7e) $\int \underbrace{e^{-x}}_{f'} \underbrace{\cos x}_{g} dx = -\underbrace{e^{-x}}_{f} \underbrace{\cos x}_{g} - \int \underbrace{(-e^{-x})}_{f}\underbrace{(-\sin x)}_{g'} dx$

$= -e^{-x}\cos x - \int \underbrace{e^{-x}}_{f'} \underbrace{\sin x}_{g} = -e^{-x}\cos x - \Big[\underbrace{(-e^{-x})}_{f}\underbrace{\sin x}_{g}$
$- \int \underbrace{(-e^{-x})}_{f} \cos x\Big]$

$= -e^{-x}\cos x + e^{-x}\sin x - \int e^{-x}\cos x + C$

$\Rightarrow 2\int e^{-x}\cos x\, dx = e^{-x}(\sin x - \cos x) + C$

$\int e^{-x}\cos x\, dx = \frac{1}{2}e^{-x}(\sin x - \cos x) + C'$